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HEAT TRANSFER FOR LAMINAR FLOW IN DUCTS WITH ARBITRARY TIME VARIATIONS IN WALL TEMPERATURE

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ABSTRACT

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An analysis is made for laminar forced convection heat transfer in a circular tube or a parallel plate channel whose walls may undergo arbitrary time variations in temperature. The time varying process can begin from an already established steady state situation with heat transfer taking place, or the fluid and walls can be initially at the same uniform temperature. The fluid velocity distribution is fully developed and unchanging with time. At any instant during the transient the wall temperature is spatially uniform, that is all portions of the wall simultaneously undergo the same temperature-time variation. The greater part of the analysis is concerned with the response to a step change in wall temperature, and the time required to reach steady state is given for this type of transient. Then the results are generalized to apply for arbitrary variations with time.

NOMENCLATURE

a	half width of spacing between parallel plates
a_n	the ratio, β_n^2/γ_n
C_n	coefficients in the series expansion of temperature distribution in a circular tube
c_p	specific heat at constant pressure
D	tube diameter, $2r_o$
F_n	a function of X and Θ in Eq. (9)
f_n	the ratio, τ_n/σ_n
G_n	a function of X and Θ in Eq. (35)
k	thermal conductivity
Pr	Prandtl number, $c_p\mu/k = \nu/\alpha$

q	local heat addition per unit area at channel wall
R	dimensionless radial coordinate, r/r_o
r	radial coordinate measured from circular tube centerline; r_o, tube radius
Re	Reynolds number; $\frac{\bar{u}D}{\nu}$ for a circular tube, $\frac{\bar{u}4a}{\nu}$ for a parallel plate channel
T	dimensionless temperature, $(t - t_o)/(t_w - t_o)$
t	temperature; t_o, temperature of fluid entering channel (a constant); t_w, wall temperature
t'	temperature difference, $t_w - t_o$
u	fluid velocity in the x-direction; $\bar{u}$, mean fluid velocity; u_{max}, maximum fluid velocity in the channel cross section
X	dimensionless axial coordinate; $\frac{x/r_o}{RePr}$ for a circular tube, $\frac{8}{3}\frac{x/a}{RePr}$ for a parallel plate channel
x	axial distance from entrance of channel
Y	dimensionless coordinate, y/a
y	normal coordinate measured from centerline of parallel plate channel
α	thermal diffusivity, $k/\rho c_p$
β_n^2	steady-state eigenvalues for circular tube
γ_n	exponential constants in transient solution for circular tube, Eq. (17)
Θ	dimensionless time; $\frac{\theta\nu}{r_o^2 Pr}$ for circular tube, $\frac{\theta\nu}{a^2 Pr}$ for parallel plate channel
θ	time
Θ^*	dummy integration variable
λ_n^2	steady-state eigenvalues for parallel plate channel
μ	absolute viscosity
ν	kinematic viscosity
ρ	fluid density
σ_n	exponential constants in solution for parallel plate channel, Eq. (36a)
τ_n	exponential constants in solution for parallel plate channel, Eq. (36b)

φ_n eigenfunctions for solution in circular tube

χ function of X in steady-state solution for circular tube, Eq. (5)

Subscripts:

s refers to steady-state condition

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INTRODUCTION

The behavior of heat transfer equipment during transient temperature changes has, in recent years, received greater attention especially in connection with the use of nuclear reactors as power sources. Since nuclear properties are often temperature dependent, it is sometimes necessary to consider in detail the thermal transients within the system, to be assured that proper control will be maintained during power changes. The present paper is concerned with studying the heat transfer behavior associated with a thermal transient in a forced convection channel flow.

Two geometries which are commonly encountered in practice are selected for analysis, the circular tube, and the parallel plate channel. The flow in the tube or between the parallel plates is assumed to be laminar, incompressible, and fully developed. The last assumption implies that a hydrodynamic entrance length is present which allows the flow to establish a fully developed velocity distribution before reaching the heated section of the channel. The transient heating process is such that the wall temperature can be specified to have an arbitrary time variation. Initially the system can be either at steady state with heat transfer taking place, or the whole system can be initially isothermal with the fluid and bounding walls all at the same temperature. At any instant of time the wall temperature is uniform, that is, all positions along the wall simultaneously undergo the same temperature-time variation.*

Most previous analyses of transient forced convection heat transfer in passages have been treated by one-dimensional approximations, that is, velocity and temperature

*For the parallel plate channel both walls are at the same temperature.

variations over the channel cross section have been neglected. A good review of work using this approach can be found for example, in references (1) and (2). The present work employs a two-dimensional analysis and thus includes the variations in velocity and temperature over the cross sections of the flow channels.

The transient heat transfer situation considered here has been treated by another method in references (3) and (4). However, in these references only the thermal entrance region is examined and the results do not extend far down the passage. The present method yields results for the entire length of the channel, but in the series expansion method which is employed, many terms are required to calculate results for the region very close to the tube entry. Thus, by joining the present results to those of references (3) and (4), information is obtained for all positions in the flow passage.

Two methods for performing the analysis are given here; one method is used in the calculations for the circular tube while the other is used for the parallel plate configuration. As will be shown in the analysis these two methods differ in the way certain required functions are computed. For the circular tube the computation involves the numerical integration of an ordinary differential equation, while for the parallel plate channel the functions are obtained by a series expansion in terms of more simple functions. A method somewhat similar to the second approach has been briefly outlined in reference (5). The method given here is not an exact solution of the governing partial differential equation, but involves an approximation at one step in the analysis. The validity of this approximation is tested by comparison with exact results available for part of the solution and good agreement is obtained. For arbitrary time variations in wall temperature, results are obtained by generalizing the transient corresponding to a unit step change in wall temperature.

The analysis thus proceeds in the following order. The circular tube is analyzed for a step in wall temperature, and the results are then generalized for arbitrary time variations. Then the parallel plate configuration is treated by a slightly different method. Results of transient heat flux variations and times required to reach steady state are given graphically and compared with previous work.

STEP-CHANGE IN WALL TEMPERATURE FOR FLOW IN A CIRCULAR TUBE

This portion of the analysis is concerned with fully developed laminar flow through a circular tube as shown in Fig. 1. The tube and fluid are assumed to be initially isothermal at temperature t_o. Then the wall is given an instantaneous step in temperature to a new value t_w and maintained at t_w for all time thereafter.

The energy equation for incompressible flow in a tube can be written as

$$\frac{\partial t}{\partial \theta} + u \frac{\partial t}{\partial x} = \alpha \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial t}{\partial r}\right) \tag{1}$$

To obtain the equation in this form viscous dissipation and axial heat conduction are neglected compared with heat conduction in the radial direction. It is convenient to rewrite this equation in terms of dimensionless variables defined as,

$$\Theta = \frac{\theta \nu}{r_o^2 Pr}; \; X = \frac{x/r_o}{RePr}; \; R = \frac{r}{r_o}; \; T = \frac{t - t_o}{t_w - t_o}$$

Θ, X, and R are respectively the dimensionless time, distance along the tube, and radial distance, and are the three independent variables of the problem. The dimensionless temperature T is defined so that at the entrance to the heated section the value of T is zero while at the tube wall T becomes unity. Since the flow is fully developed the velocity distribution has the parabolic form,

$$\frac{u}{\bar{u}} = 2\left[1 - \left(\frac{r}{r_o}\right)^2\right] \tag{2}$$

Substituting these quantities into Eq. (1) results in the following dimensionless equation,

$$\frac{\partial T}{\partial \Theta} + (1 - R^2) \frac{\partial T}{\partial X} = \frac{1}{R} \frac{\partial}{\partial R}\left(R \frac{\partial T}{\partial R}\right) \tag{3}$$

This must be solved subject to the following boundary conditions:

$$\left.\begin{array}{ll} T = 0 \quad \text{at} \quad X = 0 \quad \text{for all} \quad R \quad \text{and} \quad \Theta, & \text{entrance condition} \\ T = 0 \quad \text{at} \quad \Theta = 0, \ \text{for all} \quad R \quad \text{and} \quad X, & \text{initial condition} \\ \dfrac{\partial T}{\partial R} = 0 \quad \text{at} \quad R = 0, \qquad \text{symmetry} & \\ T = 1 \quad \text{at} \quad R = 1 \quad \text{for} \quad \text{all} \quad X \quad \text{and for} \quad \Theta > 0, & \text{specified wall temperature} \end{array}\right\} \tag{3a}$$

To obtain a solution we first consider the results for the steady-state heat transfer condition.

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Steady-state solution. - At steady state there are of course no variations with time and hence Eq. (3) reduces to

$$(1 - R^2) \frac{\partial T_s}{\partial x} = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial T_s}{\partial R}\right) \tag{4}$$

The steady state solution corresponding to the boundary conditions (3a) is for the situation of fluid at uniform temperature entering a pipe maintained at a different constant wall temperature. This problem was treated by Graetz in 1885 (see ref. (6), p. 451), and for convenience will be briefly reviewed here as the results will be needed later in the analysis.

A product solution is employed of the form

$$T_s = 1 - \chi(X)\varphi(R) \tag{5}$$

When this is inserted into Eq. (4) it is found that

$$\chi = e^{-\beta^2 X} \tag{6}$$

where $-\beta^2$ is the separation constant arising in the product solution. The equation for the function φ is

$$\frac{d^2\varphi}{dR^2} + \frac{1}{R} \frac{d\varphi}{dR} + \varphi \beta^2 (1 - R^2) = 0 \tag{7}$$

with the boundary conditions

$$\left.\begin{array}{l} \dfrac{d\varphi}{dR} = 0 \quad \text{at} \quad R = 0 \\ \varphi = 0 \quad \text{at} \quad R = 1 \end{array}\right\} \tag{7a}$$

Equations (7) and (7a) form an eigenvalue problem of the Sturm-Liouville type. Solutions are possible only for a discrete, though infinite, set of β values. Hence,

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the solution for T_s can be written as

$$T_s = 1 - \sum_{n=0}^{\infty} C_n \varphi_n(R) e^{-\beta_n^2 X} \tag{8}$$

where β_n^2 and φ_n are the eigenvalues and corresponding eigenfunctions of Eqs. (7) and (7a).

The coefficients C_n are evaluated from the boundary condition that $T_s = 0$ at the tube entrance $(X = 0)$. Applying this condition to Eq. (8), we find that the C_n must satisfy,

$$0 = 1 - \sum_{n=0}^{\infty} C_n \varphi_n \tag{8a}$$

From the properties of the Sturm-Liouville system it follows immediately that

$$C_n = \frac{\int_0^1 (R - R^3)\varphi_n \, dR}{\int_0^1 (R - R^3)\varphi_n^2 \, dR} \tag{8b}$$

The first five eigenvalues of Eq. (7) and values of C_n are available to very good accuracy in reference (7) and asymptotic results for higher values of β_n^2 and C_n are given in reference (8).

This completes the steady-state solution and we now turn to the transient problem.

Transient solution. - To form a transient solution, we try a series expansion about the steady state condition. With this type of formulation the transient solution will automatically converge, at large times, to exactly the steady-state result. Thus we try a transient solution of the same general form as the steady state equation (8),

$$T = 1 - \sum_{n=0}^{\infty} C_n F_n(X,\Theta)\varphi_n(R) \tag{9}$$

For large times the function F_n should, by comparison with Eq. (8), converge to

$$F_n = e^{-\beta_n^2 X} \tag{10}$$

The approximation which is now made is that the transient solution is only required to satisfy an integrated form of the energy equation, and will only exactly satisfy the equation in differential form for the limit of very large time. Multiplying Eq. (3) by R and integrating from 0 to 1 gives the integrated form,

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$$\int_0^1 R \frac{\partial T}{\partial \Theta} dR + \int_0^1 R(1 - R^2) \frac{\partial T}{\partial X} dR = \left(\frac{\partial T}{\partial R}\right)_{R=1} \tag{11}$$

The trial solution, Eq. (9), is then substituted into Eq. (11) with the result that each F_n must satisfy the relation

$$\frac{\partial F_n}{\partial \Theta} \int_0^1 R\varphi_n \, dR + \frac{\partial F_n}{\partial X} \int_0^1 (R - R^3)\varphi_n \, dR - F_n\left(\frac{\partial \varphi}{\partial R}\right)_{R=1} = 0 \tag{12}$$

This type of partial differential equation can be treated by using the method of characteristics. In this method the solution is obtained by considering the following auxiliary ordinary differential equations (see ref. (9), p. 368),

$$\frac{d\Theta}{\int_0^1 R\varphi_n \, dR} = \frac{dX}{\int_0^1 (R - R^3)\varphi_n \, dR} = \frac{dF_n}{-F_n \left.\frac{\partial \varphi}{\partial R}\right|_{R=1}} \tag{13}$$

Equating the second and third terms yields a solution for F_n as,

$$F_n = e^{-\frac{-\partial\varphi/\partial R\big|_{R=1}}{\int_0^1 (R-R^3)\varphi_n dR} X} \tag{14a}$$

To simplify this the steady state Eq. (7) is multiplied by R and integrated from R = 0 to R = 1, with the result

$$\beta_n^2 = \frac{-\partial\varphi/\partial R\big|_{R=1}}{\int_0^1 (R - R^3)\varphi_n \, dR}$$

Comparing this with the exponent in Eq. (14a) we find that

$$F_n = e^{-\beta_n^2 X} \tag{14b}$$

which is the desired form for steady-state as given by Eq. (10). In a similar fashion the first and third terms of Eq. (13) are equated which gives the solution

$$F_n = e^{-\frac{-\partial\varphi/\partial R\big|_{R=1}}{\int_0^1 R\varphi_n \, dR}\Theta} \tag{15}$$

Thus we have obtained two results, one being the steady-state solution, Eq. (14b), and the other, Eq. (15), a solution which is dependent on time only. The latter will be called the <u>initial transient</u> solution. The method of characteristics indicates that these two results can be joined along a characteristic line which passes through the origin of the X-Θ plane. This line is obtained by integrating the equation formed by the first and second terms of Eq. (13) subject to the condition that $X = 0$ when $\Theta = 0$. This yields

$$\Theta = \left[\frac{\int_0^1 R\varphi_n \, dR}{\int_0^1 (R - R^3)\varphi_n \, dR}\right] X \equiv a_n X \tag{16}$$

where the ratio of the two integrals has been denoted by a_n. Using an abbreviated notation for the constants involved in the exponentials we have for F_n

$$\left.\begin{aligned} F_n &= e^{-\gamma_n \Theta}, & \Theta &\leq a_n X \\ F_n &= e^{-\beta_n^2 X}, & \Theta &\geq a_n X \end{aligned}\right\} \tag{17}$$

where $a_n = \beta_n^2/\gamma_n$. To obtain the final form of the solution, F_n is substituted into Eq. (9) to give

$$T = 1 - \sum_{n=0}^{\infty} C_n \varphi_n(R) \begin{Bmatrix} e^{-\gamma_n \Theta}, & \Theta \leq a_n X \\ e^{-\beta_n^2 X}, & \Theta \geq a_n X \end{Bmatrix} \tag{18}$$

Now it is necessary to consider the boundary conditions as given by Eq. (3a). At $X = 0$, for any finite Θ, the condition $\Theta > a_n X$ in Eq. (18) is fulfilled, and we have (since $T = 0$ at $X = 0$)

$$0 = 1 - \sum_{n=0}^{\infty} C_n \varphi_n$$

This condition has already been satisfied according to Eq. (8a). For $\Theta = 0$, we have for any finite X the condition $\Theta < a_n X$ and Eq. (18) gives (since $T = 0$ at $\Theta = 0$)

$$0 = 1 - \sum_{n=0}^{\infty} C_n \varphi_n$$

which is again the same condition that has already been satisfied by proper evaluation of the C_n. The first two boundary conditions of Eqs. (3a) are thus satisfied by using the same C_n obtained in the steady-state solution. The last two boundary conditions are identically satisfied as they were used in determining the φ_n. Hence the solution satisfies all of the required boundary conditions, converges exactly to the steady-state condition for large times, and is approximate to the extent that it satisfies an integrated form of the energy equation. The accuracy of this approximation will be discussed a little later.

Numerical solution. - In order to compute heat-transfer quantities from the temperature distribution of Eq. (18) it is necessary to know values of C_n, φ_n, γ_n, β_n^2, and a_n. As mentioned before, some of these quantities are already available in the literature, but the values of φ_n were generally not tabulated in sufficient detail so that the integrations necessary to determine γ_n could be performed. To obtain the desired numerical information, Eq. (7) and the required integrals were programmed for numerical computation on an IBM 653 electronic digital computer. A forward integration scheme (Runge-Kutta) was employed, and quantities were evaluated for the first seven eigenvalues. These are listed in Table I along with the derivative of the eigenfunction at the wall which will be needed for the calculations in the next section.

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Space limitations preclude a presentation of the φ_n values but they are available as IBM listings.

Wall heat flux. - It is recalled that the transient heating process which yields the temperature distribution of Eq. (18) is one in which the wall is given a step function change in temperature from an initially isothermal condition with both the wall and fluid at the same temperature. It is of interest to look at the wall heat flux variation which is required during the transient to maintain the wall temperature constant. This heat flux can be evaluated from Fourier's law

$$q = k \left.\frac{\partial t}{\partial r}\right|_{r=r_o} \tag{19}$$

where q is defined as the heat added at the wall. Applying this to Eq. (18) gives the result for the wall heat flux as a function of position along the tube and time

$$\frac{qr_o}{k(t_w - t_o)} = - \sum_{n=0}^{\infty} C_n \left.\frac{d\varphi_n}{dR}\right|_{R=1} \begin{Bmatrix} e^{-\gamma_n \Theta}, & \Theta \leq a_n X \\ e^{-\beta_n^2 X}, & \Theta \geq a_n X \end{Bmatrix} \tag{20}$$

Before examining this expression in detail, the portion of it which forms the initial transient solution will be considered as it will indicate, to some extent, the accuracy of the integral approximation used in the analysis.

To better understand the initial transient process, consider the fluid which is at the entrance of the heated section at the time that the step in wall temperature occurs. After a period of time has elapsed, this fluid will have traveled a certain distance down the tube. Beyond this distance there has not been any penetration during the heating process of fluid which was originally outside the tube, and hence the heat flow process in this region has not been affected by the fact that the tube has an entrance. The behavior in this region is then that of a tube of infinite length in both directions, and for the case of uniform wall temperature there are no variations of heat transfer quantities with distance X. With no X variation the convective term in the energy equation (1) is identically zero and a pure transient heat conduction process takes place. The heat flux variation during this initial transient is obtained from Eq. (20) by considering the terms which vary with time only, that is, we observe the heat transfer process at sufficiently large X so that Θ will always be less than a_nX. The initial transient solution is then

$$\frac{qr_o}{k(t_w - t_o)} = - \sum_{n=0}^{\infty} C_n \left.\frac{d\varphi_n}{dR}\right|_{R=1} e^{-\gamma_n \Theta} \qquad (21)$$

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This expression provides the opportunity for evaluating, to some extent, the accuracy of the present method, since a direct comparison can be made with a known heat conduction transient. The exact solution for the initial transient is that resulting from suddenly changing the surface temperature of an infinitely long solid cylinder. The surface heat flux variation for this case is (ref. (6), p. 262)

$$\frac{qr_o}{k(t_w - t_o)} = 2 \sum_{n=0}^{\infty} e^{-\epsilon_n^2 \Theta} \qquad (22)$$

where ϵ_n are the zeros of the Bessel function J_o. Eq. (21) has been evaluated for a seven term approximation, and is compared with Eq. (22) on Fig. 2. The agreement is generally quite good, so the present solution not only gives good results near steady state as it is constrained to do, but is also well behaved in the initial transient region.

Returning now to Eq. (20), it is noted that the n'th term is a negative exponential in time until Θ becomes equal to a_nX. For larger Θ values the term is a constant which depends only on the value of X being considered. Thus, for the seven term approximation which has been carried out here, a graphical plot of the wall heat flux is found as the sum of seven different curves each of which is a negative exponential up to a certain break point and then becomes a constant. This is shown schematically on Fig. 3 which illustrates, in graphical form, the way in which the transient heat flux is determined at a given location along the tube. Eq. (20) has been evaluated for several different values of X and results are shown on Fig. 4.

The figure illustrates the transient process for several axial locations in the heated section.

Steady state times. - A quantity of practical importance is the time required for the wall heat transfer to reach the steady state value after a step in wall temperature is made. Since steady state is approached gradually, the steady state time, Θ_s, must be chosen somewhat arbitrarily. It is defined here as the time required for the heat flux to approach within five percent of the value reached for infinite time. Figure 5 presents Θ_s as a function of X. Also shown are the steady state times of Ref. (3) which were obtained by an approximate method applicable in the region of small X, and which join the present curve fairly well.

Two limiting lines are drawn on Fig. 5. The line falling below the present results is a lower bound on the steady state time which is found by saying that the heat transfer process cannot be stabilized at a certain location until a time of at least x/u_{max} has elapsed. The relation $\theta_s = x/u_{max}$ is equivalent, in the dimensionless system of variables, to $\Theta_s = X$. The upper line is obtained from the slug flow solution (Ref. (10)) which gives a steady state time of $\theta_s = x/\bar{u}$, or $\Theta_s = 2X$. This line shows that the slug flow solution yields steady state times which are too low for small values of X and too high for large X. Physically this is due to the fact that for small X the thermal boundary layers are thin, and the establishment of steady state in this region depends on the convection process near the wall. Since the velocities near the wall are small, it takes longer for the fluid to move from the tube entry to a given location than is indicated by the slug flow approximation. Hence in this region the slug flow solution underestimates the steady-state time. For large X the establishment of steady state is evidently more dependent on the velocities further away from the wall. This is because during the initial transient period, heat has been able to penetrate all the way across the tube and the fluid temperature near the wall has already been brought close to the wall temperature. Since the velocities in the central portion of the tube cross section are higher than the slug flow velocity, the steady state times are lower than slug flow predicts.

ARBITRARY TIME DEPENDENT WALL TEMPERATURE FOR FLOW IN A CIRCULAR TUBE

Fluid and wall both initially at temperature t_o. - In the previous section results were obtained which described the transient behavior following a step change in wall temperature. As shown in Fig. 6 an arbitrary wall temperature variation can be visualized as a series of differential steps, and due to the linearity of the energy equation the effects of these steps can be added to determine the response for an arbitrary variation. Since the wall temperature variation is specified, it is the wall heat flux response which must be determined.

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First consider a process in which the system is initially isothermal, and the wall is then given a step in temperature dt_w at time Θ^*. From Eq. (20) the response to this differential step is

$$dq = -\frac{k}{r_o}\sum_{n=0}^{\infty} C_n \left.\frac{d\varphi_n}{dR}\right|_{R=1} \begin{Bmatrix} e^{-\gamma_n(\Theta-\Theta^*)}, & 0 < (\Theta - \Theta^*) \leq a_n X \\ e^{-\beta_n^2 X}, & (\Theta - \Theta^*) \geq a_n X \end{Bmatrix} dt' \tag{23}$$

where $t' = t_w - t_o$. This response is then integrated over the arbitrary wall temperature variation to obtain the variation in q. For a general discussion of this type of superposition procedure, the reader is referred to Ref. (11), page 403. The integration of Eq. (23) can be put in the form,

$$\frac{qr_o}{k}(\Theta,X) = \sum_{n=0}^{n=N-1} -C_n \left.\frac{d\varphi_n}{dR}\right|_{R=1}\left[e^{-\beta_n^2 X}(t')_{\Theta - a_n X} + \int_{\Theta-a_n X}^{\Theta} -\gamma_n e^{-\gamma_n(\Theta-\Theta^*)} t'(\Theta^*)d\Theta^*\right]$$

$$+ \sum_{n=N}^{\infty}\gamma_n C_n \left.\frac{d\varphi_n}{dR}\right|_{R=1}\int_0^{\Theta} e^{-\gamma_n(\Theta-\Theta^*)} t'(\Theta^*)d\Theta^* \tag{24}$$

where for a given Θ, the value of N is found from the relation

$$a_{N-1}X < \Theta \leq a_N X \tag{24a}$$

and for $N = 0$ we define the first summation as $\sum_{n=0}^{-1} \equiv 0$, and also let $a_{-1} \equiv 0$.

To use this relation we can think in terms of evaluating the heat flux as a function of time at a particular X, say $X = X_i$. For early times, Θ will be less than all of the $a_n X_i$ and we use only the second summation from $n = 0$ to $n = \infty$. As we go to later times more and more terms are evaluated from the first summation. Since the problem, as given here, has only been evaluated for seven terms, when the time is sufficiently large so that $\Theta > a_6 X_1$ only the first summation is used.

Initial steady state heat transfer with $t_w \neq t_o$. - Here we are concerned with computing the heat transfer behavior when the transient begins from an initial steady-state heat-transfer situation. This can be done by utilizing the results of the previous section in the proper way. In the previous section the transient began with $t_w = t_o$ and we now begin again with this condition. Then we allow the wall temperature to go through any convenient transient (e.g., a step function) which will bring it to t_w, and it is kept at t_w until steady steady-state is reached. Then the specified transient is initiated and the results for this part of the computation yields the desired response from an initial steady-state heat transfer condition.

STEP CHANGE IN WALL TEMPERATURE FOR FLOW IN A PARALLEL PLATE CHANNEL

The transient problem will now be discussed for fully-developed laminar flow between parallel flat plates as illustrated in Fig. 7. We consider the same step function transient which was discussed for the circular tube, that is, the system is initially isothermal at temperature t_o and the wall temperature is then suddenly changed to t_w. The final results are in the same form as those for the circular tube case. Hence they can be generalized to include arbitrary time variations in wall temperature which yields an expression of the same form as Eq. (24). Due to this similarity only the results for a step change will be given here.

The energy equation analogous to Eq. (1) is

$$\frac{\partial t}{\partial \theta} + u\frac{\partial t}{\partial x} = \alpha \frac{\partial^2 t}{\partial y^2} \tag{25}$$

This is rewritten in dimensionless form by defining the following variables,

$$\Theta = \frac{\theta \nu}{a^2 Pr}; \qquad X = \frac{8}{3}\frac{x/a}{RePr}; \qquad Y = \frac{y}{a}; \qquad T = \frac{t - t_o}{t_w - t_o}$$

where $Re = \bar{u}4a/\nu$. Inserting the fully developed velocity distribution,

$$\frac{u}{\bar{u}} = \frac{3}{2}\left[1 - \left(\frac{y}{a}\right)^2\right] \tag{26}$$

into Eq. (25) yields the differential equation

$$\frac{\partial T}{\partial \Theta} + (1 - Y^2) \frac{\partial T}{\partial X} = \frac{\partial^2 T}{\partial Y^2} \tag{27}$$

which must be solved subject to the same boundary conditions as given by Eqs. (3a) with the coordinate Y replacing R.

<u>Steady-state solution</u>. - To obtain the transient solution we first consider the steady-state equation,

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$$(1 - Y^2) \frac{\partial T_s}{\partial X} = \frac{\partial^2 T_s}{\partial Y^2} \tag{28}$$

For the circular tube the eigenfunctions, $\varphi(R)$, of Eq. (7) were determined by numerical integration of the governing differential equation. For the parallel plate channel another approach will be illustrated which provides an alternate method of attack. To begin this method we look at the slug flow problem corresponding to the situation described by Eq. (28). For slug flow the velocity distribution, $1 - Y^2$, is replaced by a constant value. Reference (10) shows that the eigenfunctions for this case are in the form of cosines, $\cos \frac{(2m + 1)\pi Y}{2}$, where the m are integers from zero to infinity. It is then proposed to bring in the Y dependence of the velocity distribution by expanding the eigenfunctions in the solution of Eq. (28) in terms of the eigenfunctions of the slug flow problem. Thus, a solution to Eq. (28) is tried in the form

$$T_s = 1 - \sum_{n=0}^{\infty} e^{-\lambda_n^2 X} \left[b_{n0} \cos \frac{\pi Y}{2} + b_{n1} \cos \frac{3\pi Y}{2} + b_{n2} \cos \frac{5\pi Y}{2} + \ldots \right]$$

or

$$T_s = 1 - \sum_{n=0}^{\infty} e^{-\lambda_n^2 X} \left[\sum_{m=0}^{\infty} b_{nm} \cos \frac{(2m + 1)\pi Y}{2} \right] \tag{29}$$

For purposes of illustration, the procedure will be discussed here for a five term approximation, that is the m and n indices will vary from 0 to 4.

If Eq. (29) is substituted into Eq. (28) there is obtained after rearrangement

$$(1 - Y^2)\left[b_{n0} \cos \frac{\pi Y}{2} + b_{n1} \cos \frac{3\pi Y}{2} + \ldots + b_{n4} \cos \frac{9\pi Y}{2}\right]$$

$$- \left[\left(\frac{\pi}{2\lambda_n}\right)^2 b_{n0} \cos \frac{\pi Y}{2} + \left(\frac{3\pi}{2\lambda_n}\right)^2 b_{n1} \cos \frac{3\pi Y}{2} + \ldots + \left(\frac{9\pi}{2\lambda_n}\right)^2 b_{n4} \cos \frac{9\pi Y}{2}\right] = 0 \quad (30)$$

Equation (30) is then multiplied by $\cos \frac{\pi Y}{2}$ and integrated over Y from zero to one. This results in the first of the five simultaneous equations shown below. Then Eq. (30) is multiplied by $\cos \frac{3\pi Y}{2}$ and integrated to yield the second equation, and this process is repeated for each of the cosine harmonics. This yields the following set of five simultaneous equations for the five term approximation being considered

$$\left.\begin{aligned}
&\left(-\frac{\pi^2}{3} - 1 + \frac{\pi^4}{8\lambda_n^2}\right) b_{n0} - \frac{3}{4} b_{n1} + \frac{5}{36} b_{n2} - \frac{7}{144} b_{n3} + \frac{9}{400} b_{n4} = 0 \\
&-\frac{3}{4} b_{n0} + \left(-\frac{\pi^2}{3} - \frac{1}{9} + \frac{9\pi^4}{8\lambda_n^2}\right) b_{n1} - \frac{15}{16} b_{n2} + \frac{21}{100} b_{n3} - \frac{1}{12} b_{n4} = 0 \\
&\frac{5}{36} b_{n0} - \frac{15}{16} b_{n1} + \left(-\frac{\pi^2}{3} - \frac{1}{25} + \frac{25\pi^4}{8\lambda_n^2}\right) b_{n2} - \frac{35}{36} b_{n3} + \frac{45}{196} b_{n4} = 0 \\
&-\frac{7}{144} b_{n0} + \frac{21}{100} b_{n1} - \frac{35}{36} b_{n2} + \left(-\frac{\pi^2}{3} - \frac{1}{49} + \frac{49\pi^4}{8\lambda_n^2}\right) b_{n3} - \frac{63}{64} b_{n4} = 0 \\
&\frac{9}{400} b_{n0} - \frac{1}{12} b_{n1} + \frac{45}{196} b_{n2} - \frac{63}{64} b_{n3} + \left(-\frac{\pi^2}{3} - \frac{1}{81} + \frac{81\pi^4}{8\lambda_n^2}\right) b_{n4} = 0
\end{aligned}\right\} \quad (31)$$

Since these equations are homogeneous, the b_{nm} can only be nontrivial if the matrix of their coefficients is zero, and hence the λ_n^2 are the roots of the coefficient matrix. The λ_n^2 were found by solving the matrix on an IBM 653 electronic digital computer using a double precision routine which carried 16 significant figures. An accuracy of about 12 figures in the λ_n^2 values was found necessary to obtain good values of the b_{nm}. The λ_n^2 are listed in Table II and the first four show good agreement with those evaluated by other methods in Refs. (8) and (12). The poor agreement of λ_4^2 is due to the fact that only a five term approximation is being used here.

With the λ_n^2 known, the set of homogeneous Eqs. (31) cannot be solved to yield all the b_{nm}, but will only yield four coefficients in each equation in terms of the fifth coefficient which remains yet undetermined. Thus, if each equation in the set is divided by its b_{n0} the ratios b_{n1}/b_{n0}, b_{n2}/b_{n0}, etc., can be determined by solving any four of the resulting five simultaneous equations. The solution as given by Eq. (29) is now in the form

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$$T_s = 1 - \sum_{n=0}^{4} b_{n0} e^{-\lambda_n^2 X}\left[\cos\frac{\pi Y}{2} + \sum_{m=1}^{4} \frac{b_{nm}}{b_{n0}} \cos\frac{(2m+1)\pi Y}{2}\right] \tag{29a}$$

where the b_{nm}/b_{n0} are known.

To determine the b_{n0} we apply the boundary condition that $T_s = 0$ at $X = 0$. This yields

$$1 = \sum_{n=0}^{4} b_{n0}\left[\cos\frac{\pi Y}{2} + \sum_{m=1}^{4} \frac{b_{nm}}{b_{n0}} \cos\frac{(2m+1)}{2}\pi Y\right] \tag{32}$$

Equation (32) is now multiplied separately by each of the $\cos\frac{(2m+1)\pi Y}{2}$ and integrated over Y from zero to one. This yields, for the five term approximation, five simultaneous equations for the unknown b_{n0},

$$\sum_{n=0}^{4} b_{n0} = \frac{4}{\pi}$$

$$\left[\sum_{n=0}^{4}\left(\frac{b_{nm}}{b_{n0}}\right) b_{n0} = (-1)^m \frac{4}{(2m+1)\pi}\right]_{m=1,2,3,4} \tag{33}$$

When the b_{n0} have been determined from this set, Eq. (29a) provides the complete solution to the steady-state problem. The numerical values of the constants are given in Table II.

Transient solution. - The steady-state solution as given by Eq. (29a) is seen to be of the same form as that obtained for the round pipe, Eq. (8), except that the eigenfunctions are given in terms of cosine series expansions rather than by $\varphi_n(r)$ which are obtained by numerical integration of an ordinary differential equation.

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Since the two results are of the same basic form, the solution of the transient equation follows as before and will only be given in brief outline here.

The energy Eq. (27) is integrated to yield,

$$\frac{\partial}{\partial\Theta}\int_0^1 T\,dY + \frac{\partial}{\partial X}\int_0^1 (1 - Y^2)T\,dY = \left.\frac{\partial T}{\partial Y}\right|_{Y=1} \tag{34}$$

A transient solution is then assumed in the form

$$T = 1 - \sum_{n=0}^{4} b_{n0}G_n(X,\Theta)\left[\cos\frac{\pi Y}{2} + \sum_{m=1}^{4}\frac{b_{nm}}{b_{n0}}\cos\frac{(2m+1)\pi Y}{2}\right] \tag{35}$$

and this is inserted into Eq. (34). After the integrations are carried out this yields the following partial differential equation for a five term approximation,

$$\frac{2}{\pi}\left(1 - \frac{1}{3}\frac{b_{n1}}{b_{n0}} + \frac{1}{5}\frac{b_{n2}}{b_{n0}} - \frac{1}{7}\frac{b_{n3}}{b_{n0}} + \frac{1}{9}\frac{b_{n4}}{b_{n0}}\right)\frac{\partial G_n}{\partial\Theta}$$

$$+ \frac{16}{\pi^3}\left(1 - \frac{1}{(3)^3}\frac{b_{n1}}{b_{n0}} + \frac{1}{(5)^3}\frac{b_{n2}}{b_{n0}} - \frac{1}{(7)^3}\frac{b_{n3}}{b_{n0}} + \frac{1}{(9)^3}\frac{b_{n4}}{b_{n0}}\right)\frac{\partial G_n}{\partial X}$$

$$+ \frac{\pi}{2}\left(1 - 3\frac{b_{n1}}{b_{n0}} + 5\frac{b_{n2}}{b_{n0}} - 7\frac{b_{n3}}{b_{n0}} + 9\frac{b_{n4}}{b_{n0}}\right)G_n = 0$$

This is of the same form as Eq. (12) and hence the solution can immediately be written by analogy with the previous solution Eq. (18),

$$T = 1 - \sum_{n=0}^{4} b_{n0}\left[\cos\frac{\pi Y}{2} + \sum_{m=1}^{4}\frac{b_{nm}}{b_{n0}}\cos\frac{(2m+1)\pi Y}{2}\right]\begin{Bmatrix} e^{-\sigma_n\Theta}, & \Theta \le f_nX \\ e^{-\tau_nX}, & \Theta \ge f_nX \end{Bmatrix} \tag{36}$$

where

$$\sigma_n = \frac{\pi^2}{4}\,\frac{\left(1 + \sum_{m=1}^{4}(2m+1)(-1)^m\frac{b_{nm}}{b_{n0}}\right)}{\left(1 + \sum_{m=1}^{4}\frac{1}{(2m+1)}(-1)^m\frac{b_{nm}}{b_{n0}}\right)} \tag{36a}$$

$$\tau_n = \frac{\pi^4}{32} \frac{\left(1 + \sum_{m=1}^{4} (2m + 1)(-1)^m \frac{b_{nm}}{b_{n0}}\right)}{\left(1 + \sum_{m=1}^{4} \frac{1}{(2m + 1)^3} (-1)^m \frac{b_{nm}}{b_{n0}}\right)} \tag{36b}$$

and

$$f_n = \tau_n/\sigma_n$$

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Numerical values of σ_n, τ_n, and f_n are given in Table II for a five term approximation. Table III gives values for a three term approximation which, when compared with Table II, indicates how the solution converges with increasing numbers of terms in the approximation.

Wall heat flux. - By using Fourier's law the heat transferred at the channel walls is obtained as

$$\frac{qa}{k(t_w - t_o)} = \frac{\pi}{2} \sum_{n=0}^{\infty} b_{n0} \left[1 + \sum_{m=1}^{\infty} \frac{b_{nm}}{b_{n0}} (-1)^m (2m + 1)\right] \left\{ \begin{array}{ll} e^{-\sigma_n \Theta}, & \Theta \leq f_n X \\ e^{-\tau_n X}, & \Theta \geq f_n X \end{array} \right\} \tag{37}$$

In the initial transient period this reduces to the purely time dependent equation,

$$\frac{qa}{k(t_w - t_o)} = \frac{\pi}{2} \sum_{n=0}^{\infty} b_{n0} \left[1 + \sum_{m=1}^{\infty} \frac{b_{nm}}{b_{n0}} (-1)^m (2m + 1)\right] e^{-\sigma_n \Theta} \tag{38}$$

This can be compared with the pure conduction solution resulting from suddenly changing the temperature at the surfaces of a slab of finite thickness and infinite extent (see ref. (6), p. 262)

$$\frac{qa}{k(t_w - t_o)} = 2 \sum_{n=0}^{\infty} e^{-(2n+1)^2 \frac{\pi^2}{4} \Theta} \tag{39}$$

Curves calculated from Eqs. (38) and (39) are included on Fig. 8, and good agreement is obtained.

Figures 8 and 9 show the transient heat flux and steady state times as a function of location along the channel. These results were obtained from Eq. (37) in the same way that the computations were made for the circular tube case. The lower limiting curve on Fig. 9 is obtained by letting the steady-state time be x/u_{max} which is equivalent to $\Theta_s = X$, while for the slug flow assumption $\theta_s = \frac{x}{u}$ or $\Theta_s = \frac{3}{2} X$. The curve of steady state time for small X, as given in Ref. (4), is also shown and joins fairly well onto the present results.

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BIBLIOGRAPHY

1. Clark, J. A., Arpaci, V. S., and Treadwell, K. M.: Dynamic Response of Heat Exchangers Having Internal Heat Sources - Part I. ASME Transactions, vol. 80, no. 3, April 1958, pp. 612-624.

2. Arpaci, V. S., and Clark, J. A.: Dynamic Response of Heat Exchangers Having Internal Heat Sources - Part II. ASME Transactions, vol. 80, no. 3, April 1958, pp. 625-634.

3. Sparrow, E. M., and Siegel, R.: Thermal Entrance Region of a Circular Tube under Transient Heating Conditions. Proc. of the Third U. S. Congress for Applied Mechanics, Brown Univ., June 1958.

4. Siegel, R., and Sparrow, E. M.: Transient Heat Transfer for Laminar Forced Convection in the Thermal Entrance Region of Flat Ducts. ASME Paper No. 58-HT-3, Presented at the ASME-AIChE Heat Transfer Conference, Chicago, Ill., August, 1958.

5. Millsaps, Knox, and Pohlhausen, Karl: Heat Transfer to Hagen-Poiseuille Flows. Proc. Conf. on Differential Equations, Univ. of Maryland, Mar. 1955.

6. Jakob, M.: Heat Transfer. Vol. I. John Wiley & Sons, Inc., 1949.

7. Lipkis, R. P.: Discussion of Paper by Sellars, Tribus, and Klein. ASME Trans., vol. 78, no. 2, February 1956, p. 448.

8. Sellars, J. R., Tribus, M., and Klein, J. S.: Heat Transfer to Laminar Flow in a Round Tube or Flat Conduit - The Graetz Problem Extended. ASME Trans., vol. 78, no. 2, February 1956, pp. 441-448.

9. Hildebrand, F. B.: Advanced Calculus for Engineers. Prentice-Hall Inc., 1949.

10. Siegel, R.: Transient Heat Transfer for Laminar Slug Flow in Ducts. (To be published in Journal of Appl. Mech.)

11. von Kármán, T., and Biot, M. A.: Mathematical Methods in Engineering. McGraw-Hill Book Co., Inc., 1940.

12. Yih, C. S., and Cermak, J. E.: Laminar Heat Convection in Pipes and Ducts. Report No. 5, Civil Engineering Department, Colorado Agricultural and Mechanical College, Sept. 1951.

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TABLE I. - COEFFICIENTS FOR FLOW IN CIRCULAR TUBE

n	β_n^2	γ_n	C_n	$d\varphi_n/dR \vert_{R=1}$	a_n
0	7.31358	5.1540	1.4764	-1.0143	1.4190
1	44.6095	16.262	-.80612	1.3492	2.7432
2	113.921	29.918	.58876	-1.5723	3.8078
3	215.241	45.392	-.47585	1.7460	4.7418
4	348.564	62.327	.40502	-1.8909	5.5925
5	513.78	80.324	-.35542	2.0145	6.3963
6	711.11	99.592	.31892	-2.1265	7.1403

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TABLE II. - COEFFICIENTS IN FIVE TERM APPROXIMATION FOR

FLOW BETWEEN PARALLEL PLATES

n	b_{n0}	b_{n1}/b_{n0}	b_{n2}/b_{n0}	b_{n3}/b_{n0}	b_{n4}/b_{n0}
1	1.17776	0.0211834	-0.00113896	0.000207037	-0.0000586707
2	.0579815	-5.37195	-.838971	.0230766	-.00930795
3	.0165696	-4.00649	9.45808	3.31526	.101483
4	.00706883	-3.59673	7.94700	-12.0113	-7.12402
5	.0138607	-3.32432	6.61879	-11.0745	13.7624

n	λ_n^2	σ_n	τ_n	f_n
1	2.82776	2.30858	2.82948	1.22564
2	32.1475	11.9441	32.3656	2.70975
3	93.4792	24.9156	95.2824	3.82421
4	187.388	37.4291	178.074	4.75764
5	414.761	92.5592	608.811	6.57754

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TABLE III. - COEFFICIENTS IN THREE TERM APPROXIMATION

FOR FLOW BETWEEN PARALLEL PLATES

n	b_{n0}	b_{n1}/b_{n0}	b_{n2}/b_{n0}
1	1.18025	0.0211847	-0.00114101
2	.0589906	-5.37097	-.842957
3	.0339989	-3.89954	8.99210

n	λ_n^2	σ_n	τ_n	f_n
1	2.82777	2.31337	2.83546	1.22568
2	32.1508	12.1389	32.9332	2.71304
3	104.683	34.7142	144.296	4.15668

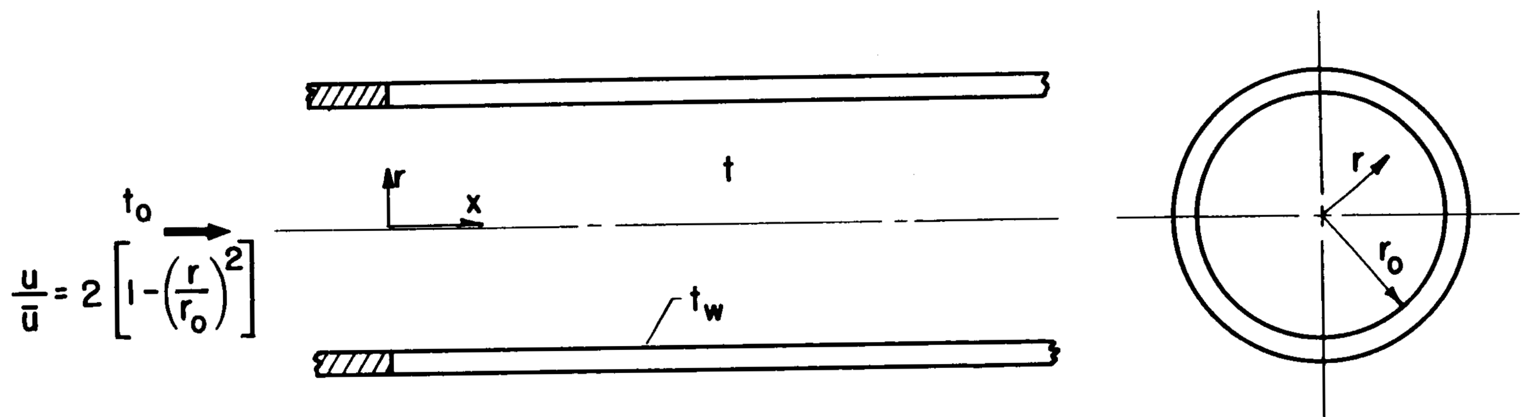

Fig. 1. - Coordinate system for circular tube geometry.

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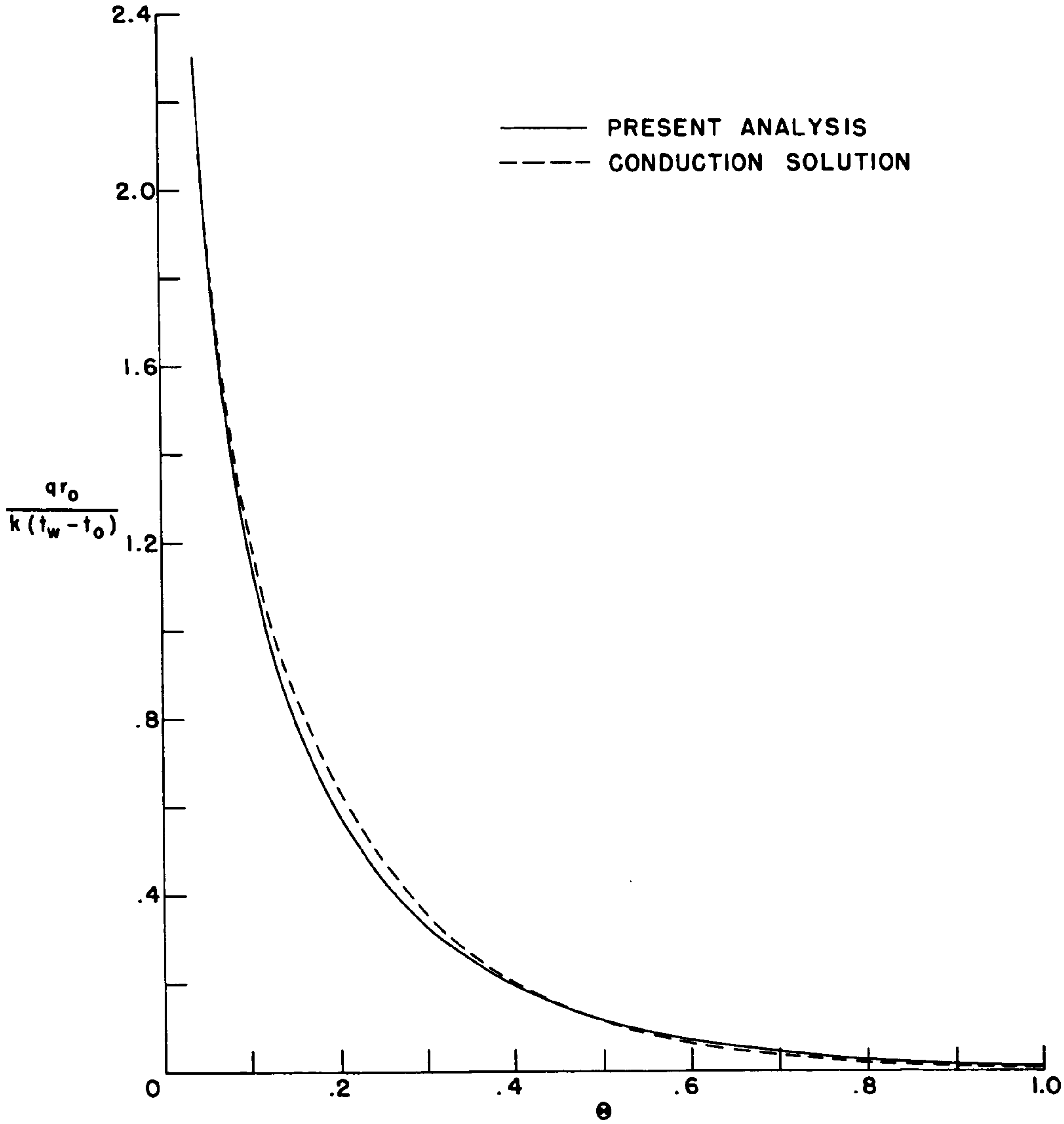

Fig. 2. - Comparison of initial transient results in round tube with conduction transient.

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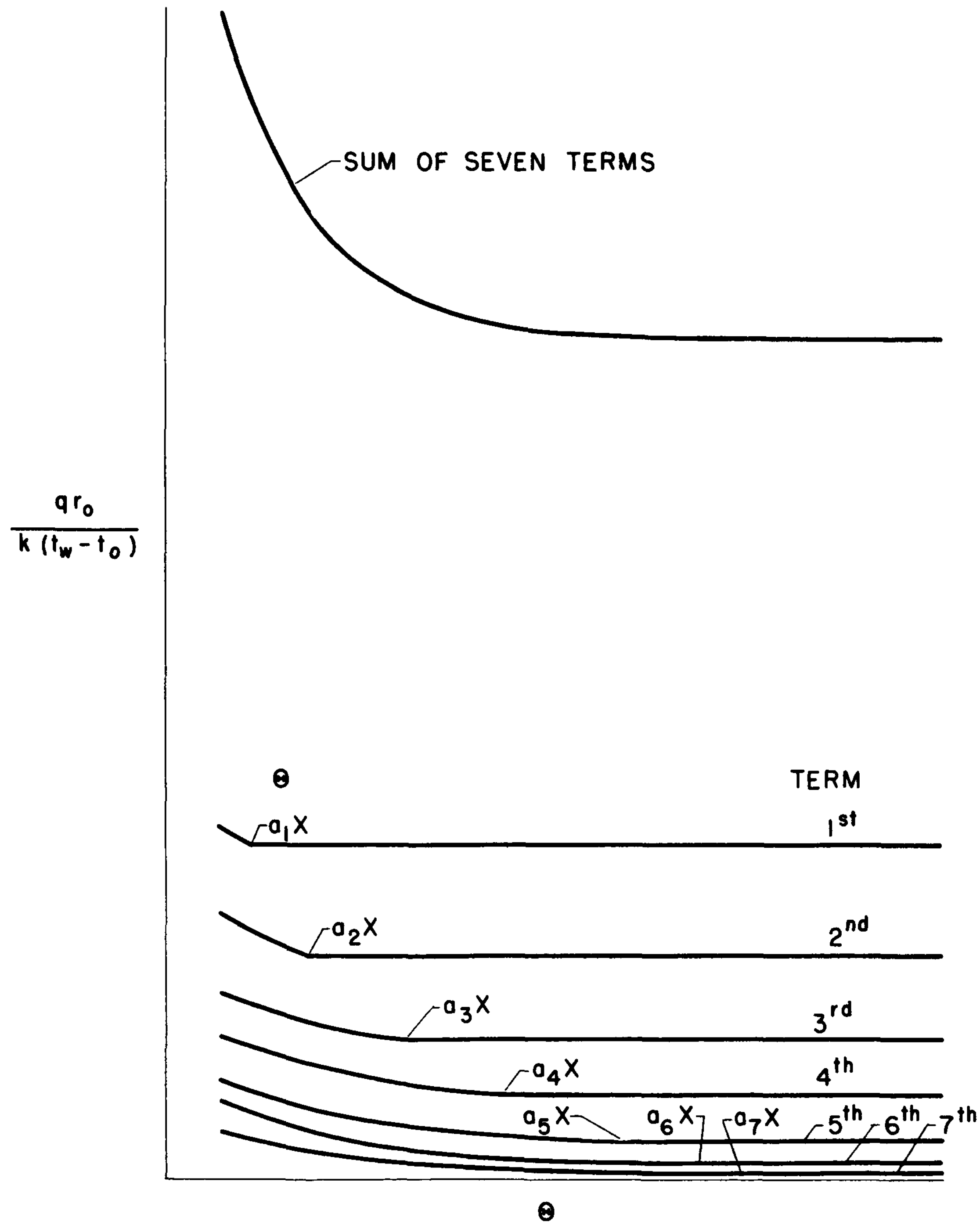

Fig. 3. - Schematic representation of wall heat flux computation following a step change in wall temperature.

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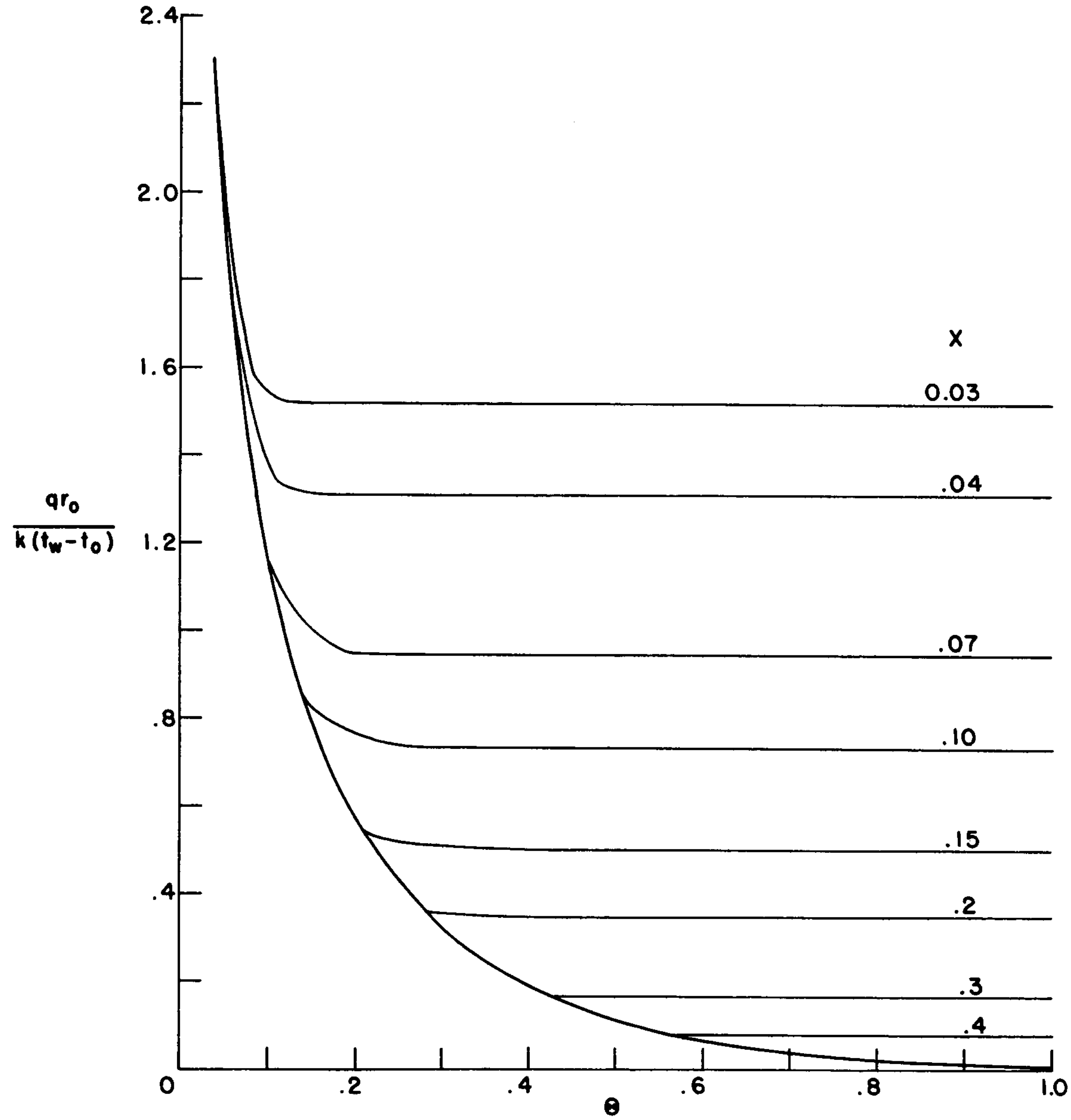

Fig. 4. - Transient variation in wall heat flux following a step change in wall temperature for flow in a circular tube.

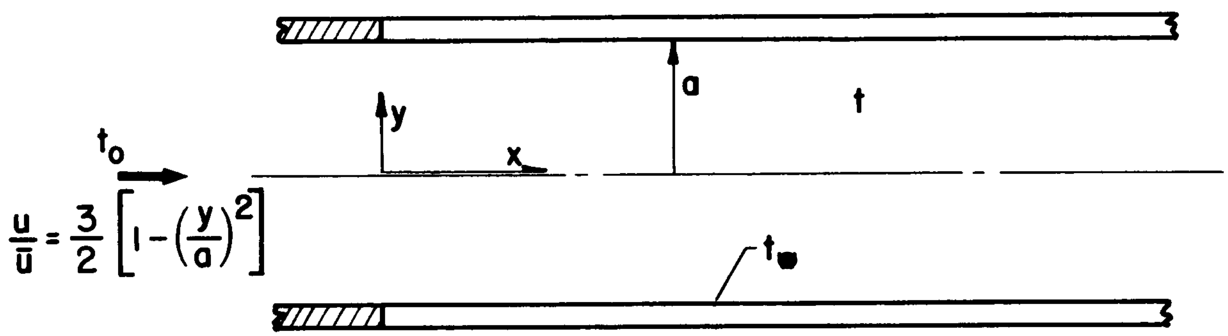

Fig. 7. - Parallel plate coordinate system.

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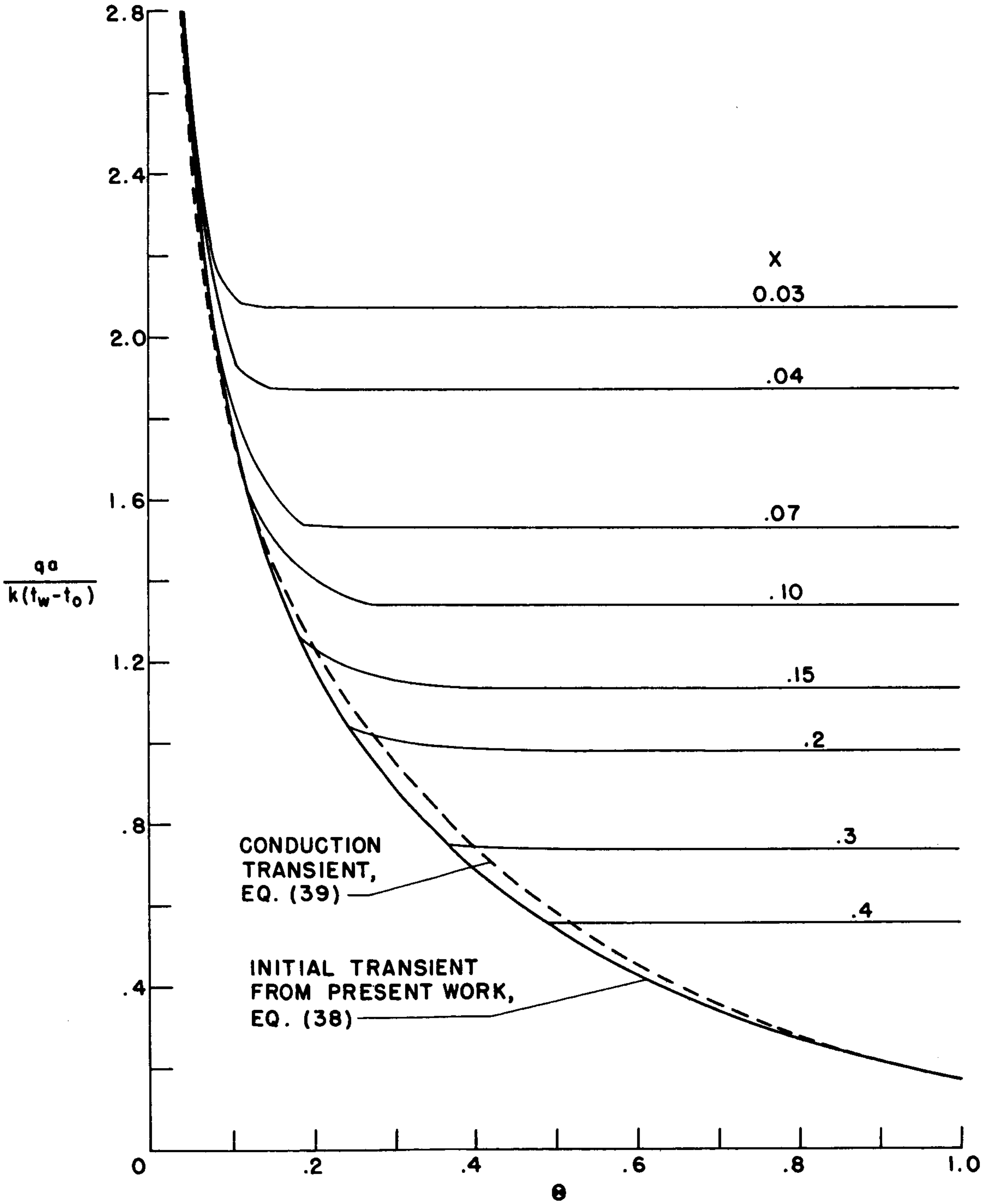

Fig. 8. - Transient variation in wall heat flux following a step change in wall temperature for flow between parallel flat plates.

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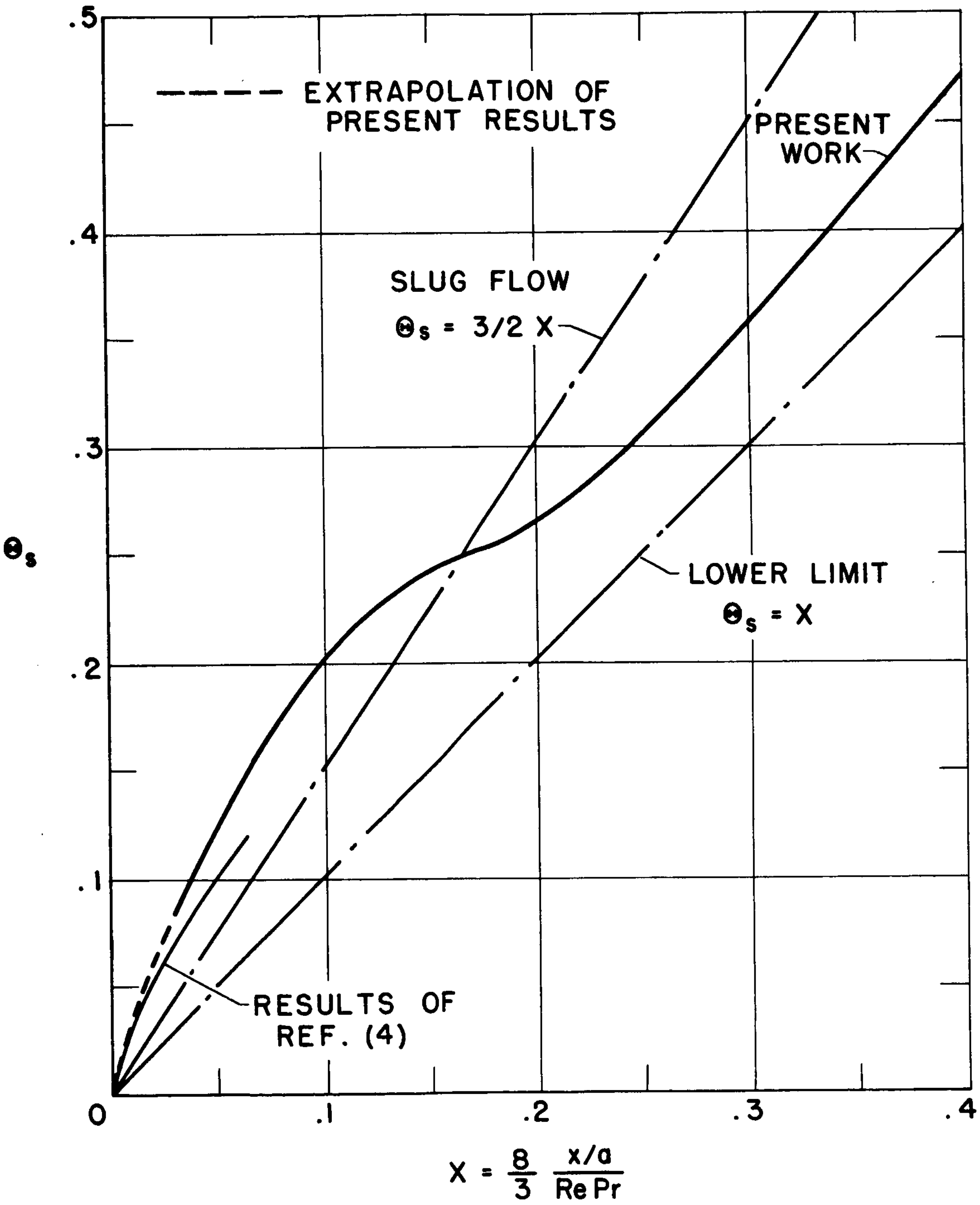

Fig. 9. - Time to reach steady state after a step change in wall temperature for flow between parallel flat plates.